

MATH/STAT 355 Algebra & Calculus Review

1 Algebra

1. Show that for $(x_n), k \geq 0$,
 - (a) $\prod_k e^{x_k} = e^{\sum_k x_k}$,
 - (b) but that $\prod_k x_k^t \neq x_k^{\sum_k t}$
2. Show that for $(x_k), k \geq 0$,
 - (a) $\log(\prod_k x_k) = \sum_k \log(x_k)$,
 - (b) but that $\log(\prod_k x_k) \neq \log(\sum_k x_k)$

2 Differentiation - Basics

1. $\frac{\partial}{\partial x}(4x^5 - 5x^4)$
2. $\frac{\partial}{\partial x}(3x^2 + 7x - 10)$
3. $\frac{\partial}{\partial y}(3x^2 + 7xy - 10y)$
4. $\frac{\partial}{\partial \alpha} \sum_{i=1}^n \alpha^2 \beta^2 x_i^2$
5. $\frac{\partial}{\partial x} x e^x$
6. $\frac{\partial}{\partial x} \alpha e^x$
7. $\frac{\partial}{\partial x} e^x \sin(x)$
8. $\frac{\partial}{\partial \alpha} \alpha e^x$
9. $\frac{\partial}{\partial x} \frac{x^2 - 1}{x}$
10. $\frac{\partial}{\partial x} \frac{\log(x)}{x}$
11. $\frac{\partial}{\partial x} \frac{(3x^2 - \pi x)^3}{10}$
12. $\frac{\partial}{\partial x} (3x + 5)^6$
13. $\frac{\partial}{\partial x} \log(\sin(x))$
14. $\frac{\partial}{\partial x} \log(x^5)$
15. $\frac{\partial}{\partial \alpha} \alpha^2 \log(\beta^5 + 3\beta + 7)$
16. $\frac{\partial}{\partial x} (2x - \frac{4}{\sqrt{x}})$
17. $\frac{\partial}{\partial \beta} \sum_{i=1}^n \log(\beta) e^{2x_i}$
18. $\frac{\partial}{\partial \alpha} \log(\prod_{i=1}^n \alpha^{x_i})$
19. $\frac{\partial}{\partial x} \sqrt{x^2 + 8}$
20. $\sum_{i=1}^n x_i y_i = \sum_{i=1}^n x_i \sum_{i=1}^n y_i$ True/False?
21. $\prod_{i=1}^n x_i^\alpha = x_i^{n\alpha}$ True/False?
22. $\sum_{i=1}^n \log(x_i) = \log(\sum_{i=1}^n x_i)$ True/False?
23. $\sum_{i=1}^n x_i \log(\alpha) = n \log(\alpha) \sum_{i=1}^n x_i$ True/False?

3 Differentiation - Matrices+

1. Show $\frac{d}{d\mathbf{x}}(\mathbf{x}^T \mathbf{B}) = \mathbf{B}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n \times m}$. Note:

$$\frac{d}{d\mathbf{x}}[\phi] := \begin{bmatrix} \frac{d\phi}{dx_1} \\ \vdots \\ \frac{d\phi}{dx_n} \end{bmatrix}$$

$$\begin{aligned} \frac{d(\mathbf{x}^T \mathbf{B})}{d\mathbf{x}} &= \frac{d}{d\mathbf{x}} \left(\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{d}{dx_1}(b_{11}x_1 + \dots + b_{n1}x_n) & \dots & \frac{d}{dx_1}(b_{1m}x_1 + \dots + b_{nm}x_{nm}) \\ \vdots & \ddots & \vdots \\ \frac{d}{dx_n}(b_{11}x_1 + \dots + b_{n1}x_n) & \dots & \frac{d}{dx_n}(b_{1m}x_1 + \dots + b_{nm}x_{nm}) \end{bmatrix} \\ &= \mathbf{B} \end{aligned}$$

2. Show $\frac{d}{d\mathbf{x}}(\mathbf{x}^T \mathbf{b}) = \mathbf{b}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$. (Solution/proof is the same as above)
3. Show $\frac{d}{d\mathbf{x}}(\mathbf{x}^T \mathbf{x}) = 2\mathbf{x}$
4. Show $\frac{d}{d\mathbf{x}}(\mathbf{x}^T \mathbf{B} \mathbf{x}) = 2\mathbf{B} \mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n \times m}$
5. Maximize $|x^m - x^n|$ on the interval $(0, 1)$. Assume $m > n$.

$$\begin{aligned} |x^m - x^n| \text{ on } (0, 1) &= (x^m - x^n) \\ \frac{d}{dx}(x^m - x^n) &= mx^{m-1} - nx^{n-1} \\ x^{m-n} &= \frac{n}{m} \\ x &= \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \end{aligned}$$

6. Challenge: Show that if $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$. (Hint: create a function that is always greater than or equal to 0, and use calculus to minimize it).

4 Limits and Integration

4.1 Limits

1. $\lim_{n \rightarrow \infty} \frac{e^n}{n!}$

$$\lim_{n \rightarrow \infty} \frac{e^n}{n!} = 0$$

Because $e \cdot e \cdot \dots \cdot e < 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ for sufficiently large n .

2. $\lim_{x \rightarrow \infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \alpha \in \mathbb{Z}$

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \\ &= \lim_{x \rightarrow \infty} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{(\alpha-1)x^{\alpha-2}}{e^{\beta x}} \quad (\text{by l'Hôpital}) \\ & \vdots \\ &= \lim_{x \rightarrow \infty} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{(\alpha-1)!}{\beta^\alpha e^{\beta x}} \quad (\text{by l'Hôpital}) \\ &= \lim_{x \rightarrow \infty} \frac{(\alpha-1)!}{(\alpha-1)!} e^{-\beta x} \\ &= 0 \end{aligned}$$

3. $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$

4.2 Integration

1. $\int_{-\infty}^{\infty} e^{-x^2} dx$

Consider

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-x^2} \right)^2 &= \int_{-\infty}^{\infty} e^{-x^2} \int_{-\infty}^{\infty} e^{-y^2} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \\ &= 2\pi \int_0^{\infty} e^{-r^2} r dr d\theta \end{aligned}$$

Let $u = r^2$ so $\frac{1}{2} du = r dr$, and so

$$\begin{aligned} &= \pi \int_0^{\infty} e^{-u} du \\ &= \pi [-e^{-u}]_0^{\infty} \\ &= \pi(1) \end{aligned}$$

which implies $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

2. $\int_1^e \int_0^1 x^{y-1} dx dy$

3. $\int_{-1}^1 \int_{-1}^1 e^y x^3 dx dy$

4. $\int \ln(x) dx$

5. Prove $z\Gamma(z) = \Gamma(z+1)$, where

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt$$

$$\begin{aligned} z\Gamma(z) &= \int_0^{\infty} z t^{z-1} e^{-t} dt \\ &= t^z e^{-t} \Big|_0^{\infty} + \int_0^{\infty} t^{(z+1)-1} e^{-t} dt \\ &= 0 + \Gamma(z+1) \\ &= \Gamma(z+1) \end{aligned}$$